

# 산사태 위험도 평가모형 (TiVaSS) 및 토석류 모형 (Deb2D) 개발

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CHUNGNAM NATIONAL UNIVERSITY

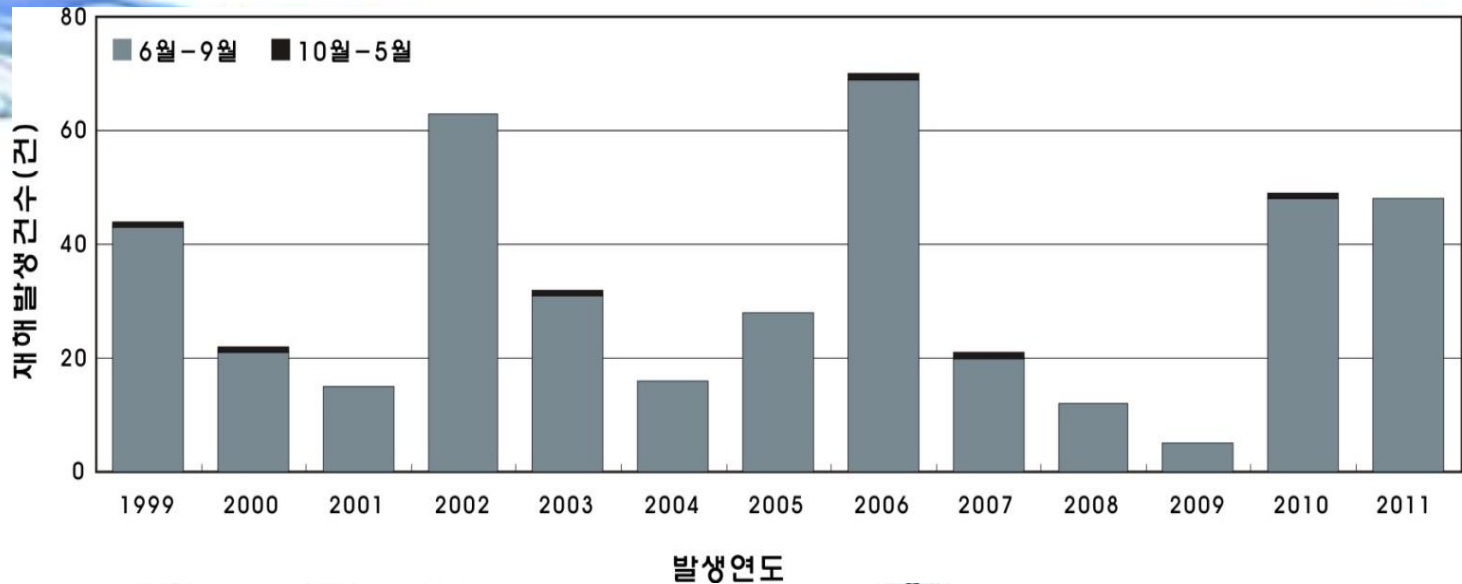
# Background



Earthquake

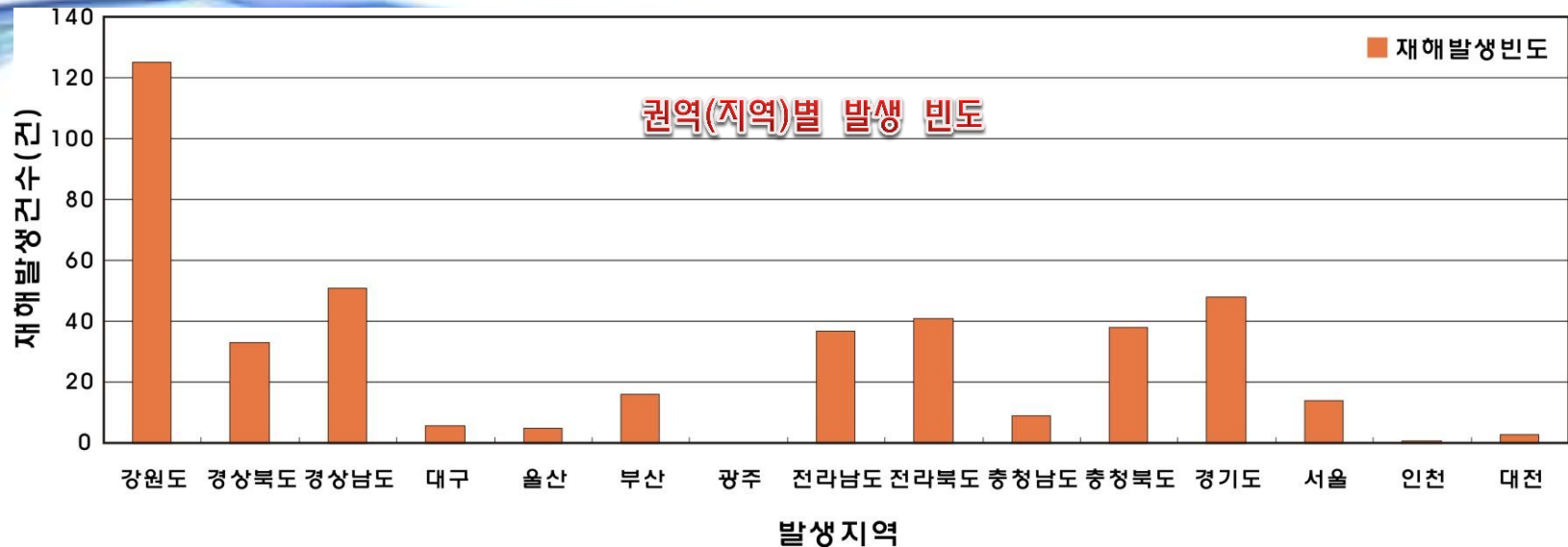
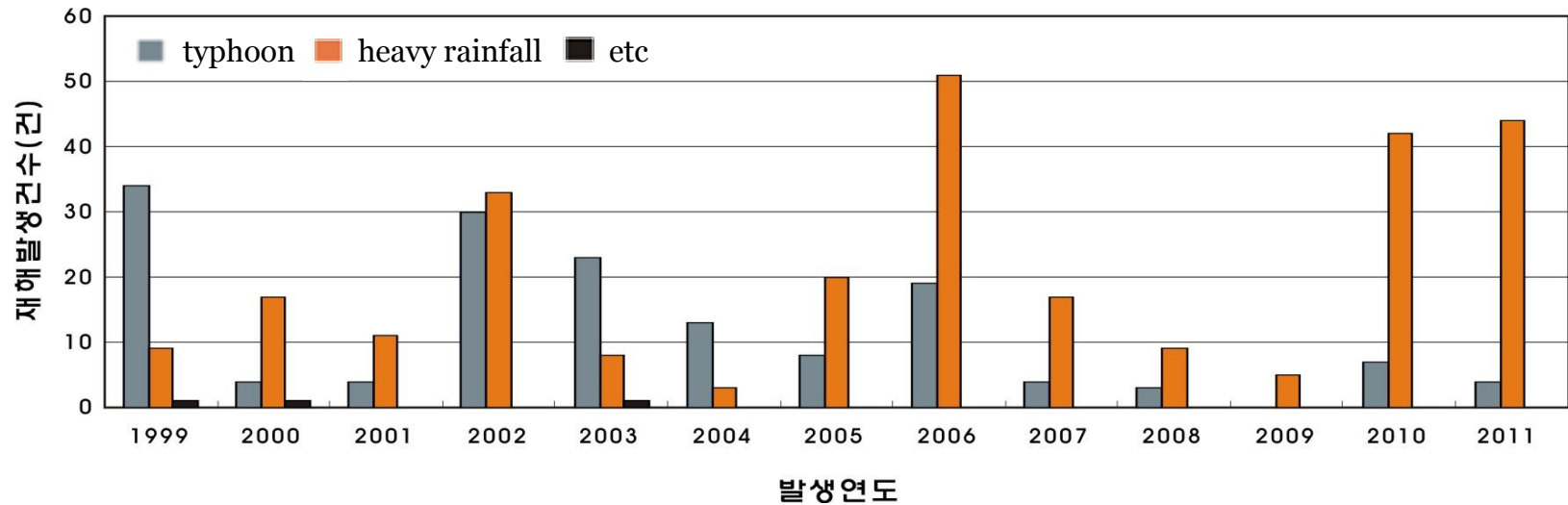
Rainfall

급경사지 재해의 대부분은  
6-9월(우기)에 집중됨



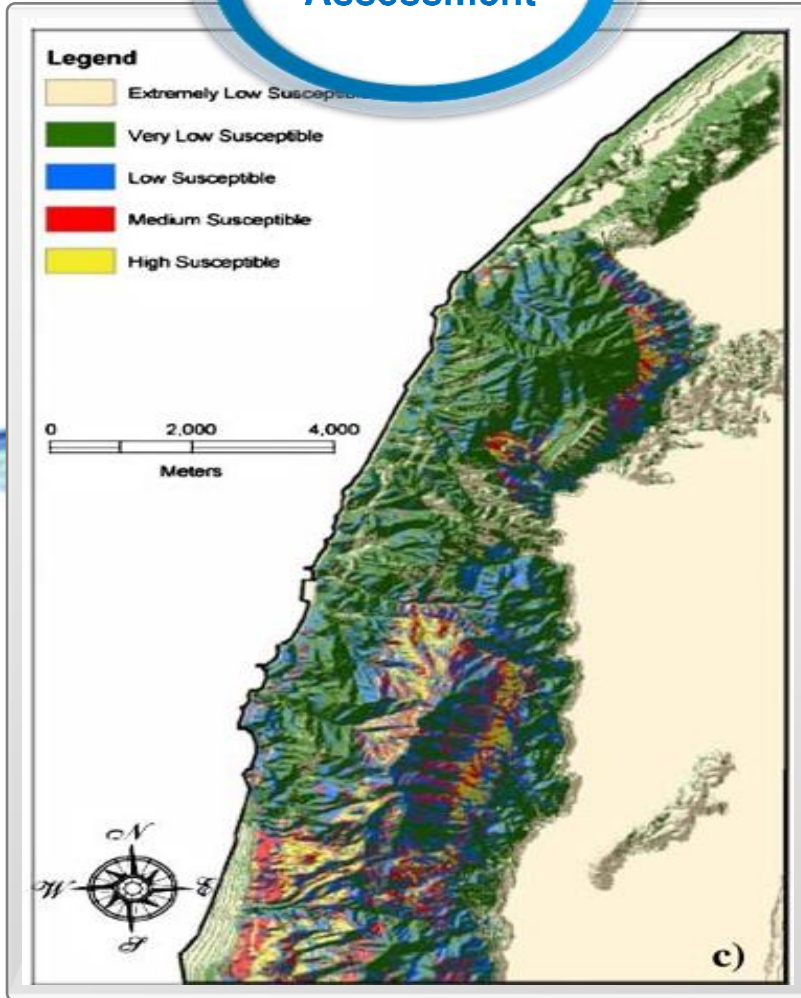
출처: 2012년 자연재해저감기술개발사업, 집중호우를 고려한 급경사지 재해위험도 정밀평가기법 및 기반재해위험지도 개발 발표자료

# Background

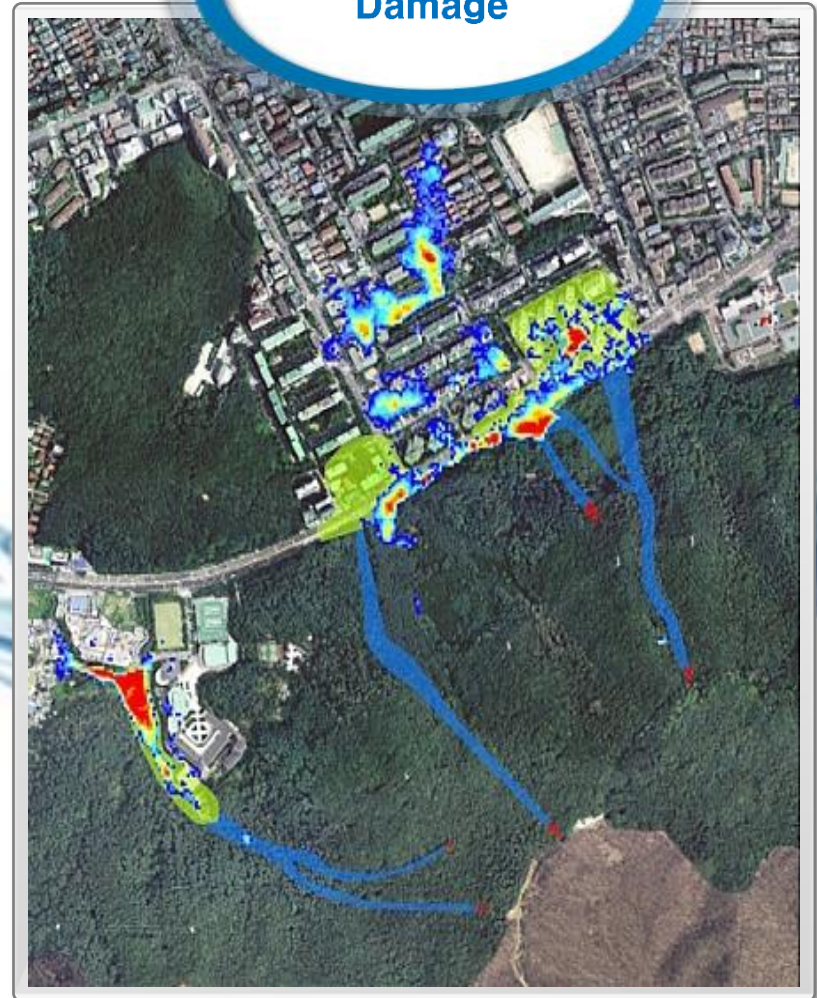


# Topics

## Landslide Risk Assessment



## Debris Flow Damage



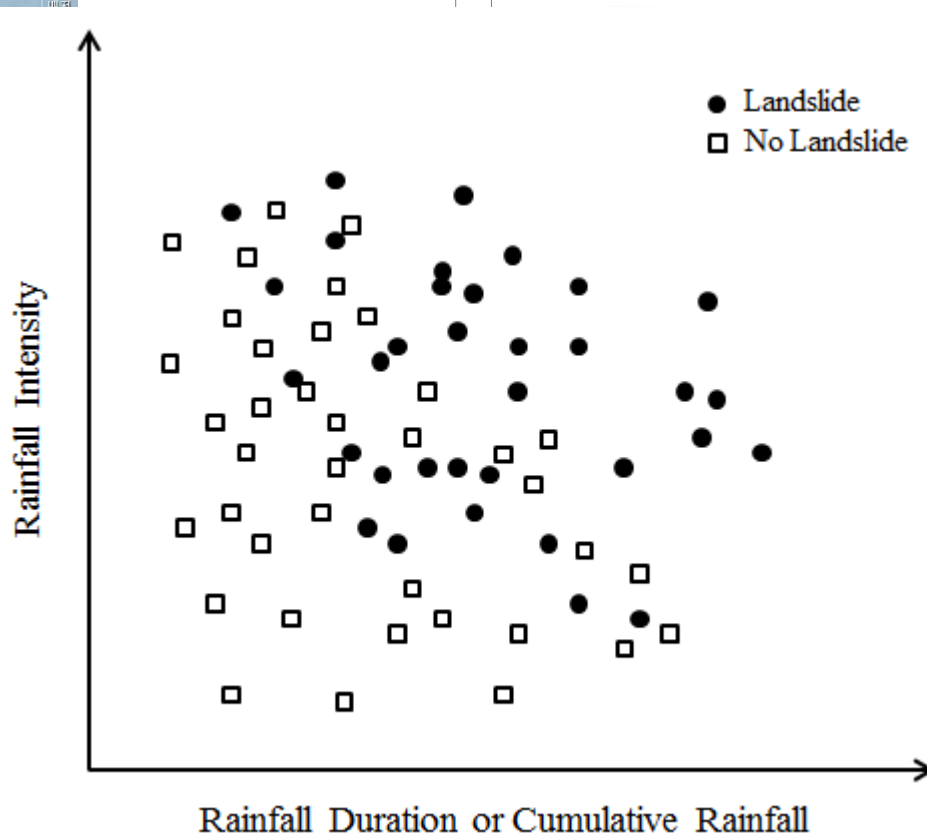
# Landslide Risk Assessment method

## Empirical Method

구분	배점	구분	배점	
붕괴 위험성	경사길이	20	붕괴	임상
	사면높이	20		사면형
	강우강도	20		토질
	연간강우량	20	위험성	경사도
	지속강우일	20	사외적 영향성	주변환경(구조물및
	붕괴 이력	20		인가호수(가
	모암	20		인가-공공시설 등
사면 내 붕괴예상범위	20			

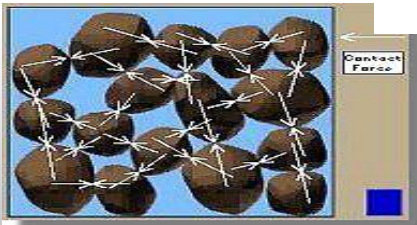
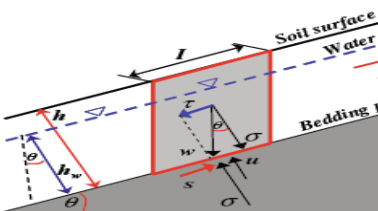
- 붕괴위험도 등급  
 A(0~100) : 붕괴위험성 없음  
 B(101~150) : 붕괴위험성 거의 없음  
 C(151~200) : 붕괴위험성 있음  
 D(201~250) : 붕괴위험성 높음  
 E(251이상) : 붕괴위험성 매우 높음

## Process-based Method



- 위험요소와 붕괴 사이의 통계적 분석을 통해 위험도를 평가
- 이변량 or 다변량 통계 분석
- logistic regression, ANN, GA, etc

## Deterministic

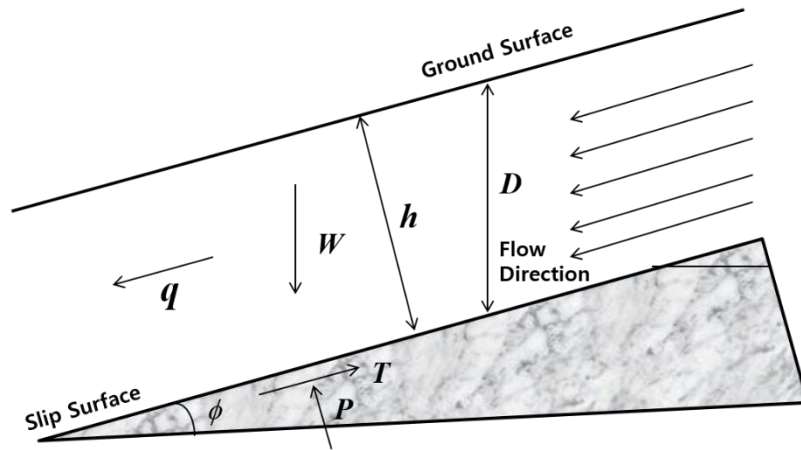


- 사면의 붕괴메커니즘을 고려
- 자료의 불확실성(?)

Decreasing reliability of landslide warning



# Deterministic (Physical) Method



Factor of Safety

$$FS = -\frac{\psi S_e \gamma_w \tan \phi}{\gamma_s D \sin \phi \cos \phi} + \frac{\tan \phi}{\tan \phi} + \frac{c}{\gamma_s D \sin \phi \cos \phi}$$

$\psi$  : pore water pressure head (m)

**Subsurface water movement**  
- Richards Eq.

$$\frac{\partial \theta}{\partial t} - \nabla \cdot (K \nabla (\psi + z)) = 0$$

**3D-model**

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z} \left( K \frac{\partial (\psi + z)}{\partial z} \right) = 0$$

**1D-model**

# Deterministic (Physical) Method

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z} \left( K \frac{\partial (\psi + z)}{\partial z} \right) = 0$$

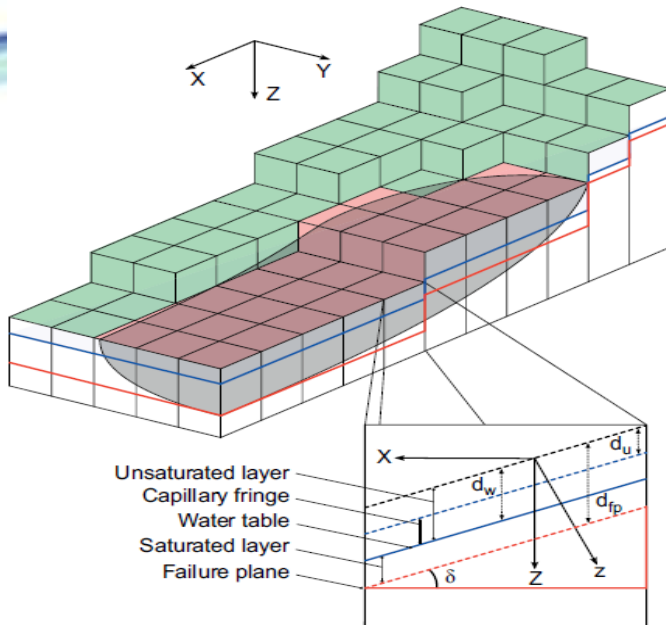
Assumption & linearization

1D analytical solution



**TRIGRS**

(Iverson, 2000; Baum et al., 2008)



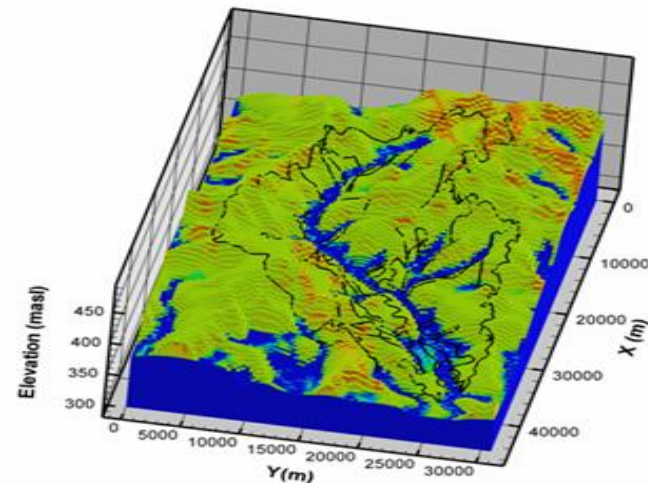
$$\frac{\partial \theta}{\partial t} - \nabla \cdot (K \nabla (\psi + z)) = 0$$

Numerical method

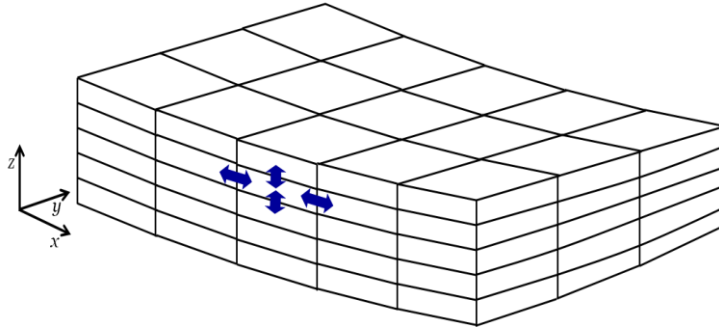
3D numerical solution



**Target**



# Numerical solver of 3D subsurface flow



## Assumption

- The effect of horizontal flow is not significant as compared with vertical flow

Usual discretization (full implicit, backward Euler)

$$\frac{\theta^{n+1} - \theta^n}{\Delta t} - \frac{\partial}{\partial x} \left( K \frac{\partial \psi^{n+1}}{\partial x} \right) - \frac{\partial}{\partial y} \left( K \frac{\partial \psi^{n+1}}{\partial y} \right) - \frac{\partial}{\partial z} \left( K \frac{\partial (\psi^{n+1} + z)}{\partial z} \right) = 0$$

Our approach (partially implicit)

$$\frac{\theta^{n+1} - \theta^n}{\Delta t} - \frac{\partial}{\partial x} \left( K \frac{\partial \psi^{n*}}{\partial x} \right) - \frac{\partial}{\partial y} \left( K \frac{\partial \psi^{n*}}{\partial y} \right) - \frac{\partial}{\partial z} \left( K \frac{\partial (\psi^{n+1} + z)}{\partial z} \right) = 0$$

Horizontal flux is computed by pre-determined interval

**much less CPU time**



# GUI development



The image shows the Qt Designer interface for GUI development. The central canvas displays a form with two validators: a QIntValidator and a QDoubleValidator. The QIntValidator has a minimum value of 0 and a maximum of 1000. The QDoubleValidator has a minimum of 0.00, a maximum of 1000.00, a format of Standard, and 2 decimal places. A 'Quit' button is located at the bottom right of the form.

On the left, the 'Design' palette shows various widget types such as Spacers, Buttons, and Containers. The 'Signals & Slots Editor' at the bottom shows a connection between a 'pushButton' and a 'Valid...sForm' object, with the signal 'clicked()' and the slot 'close()'.

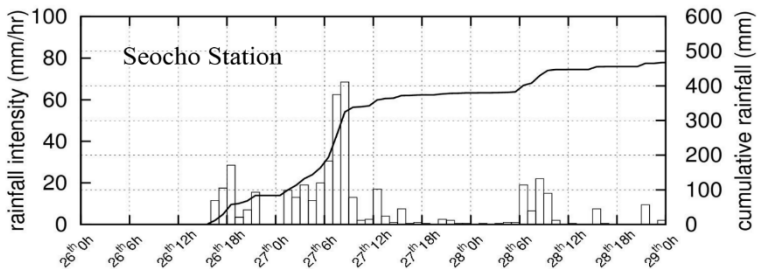
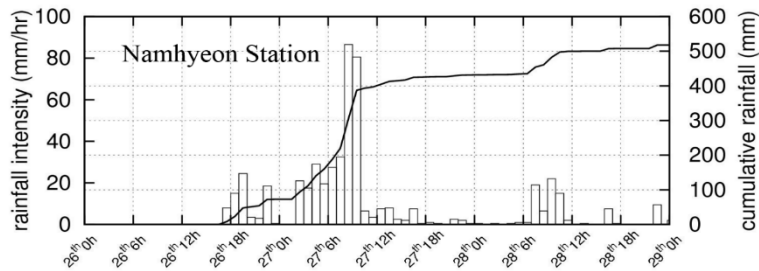
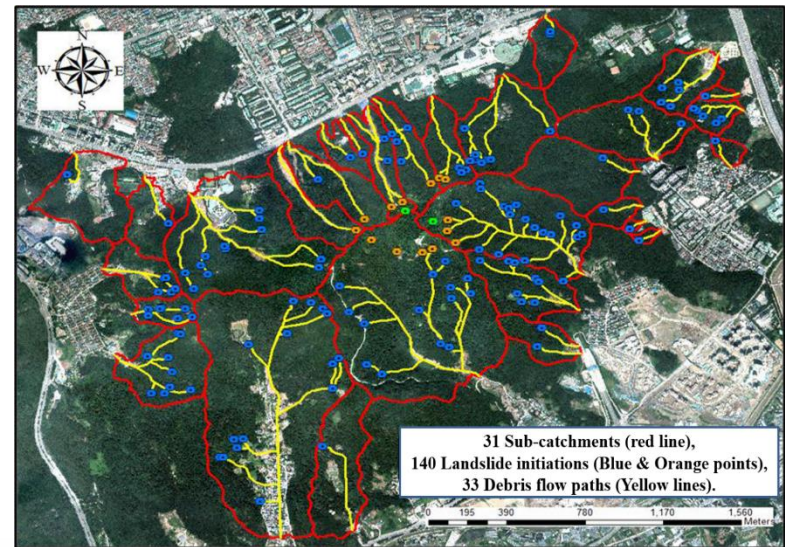
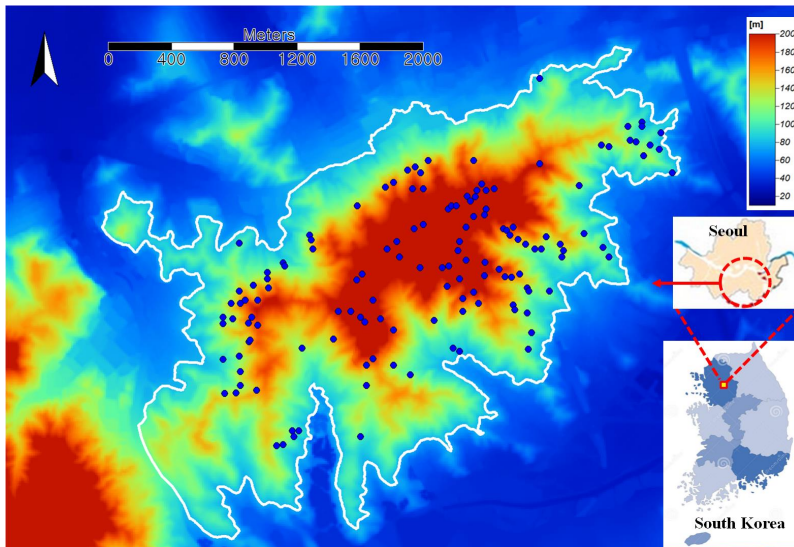
On the right, the 'Object Inspector' shows a tree view of the widget hierarchy. The 'Property Inspector' below it shows the properties of the selected 'ValidatorsForm' widget, including its geometry and size policy.

Sender	Signal	Receiver	Slot
pushButton	clicked()	Valid...sForm	close()

Property	Value
objectName	ValidatorsForm
enabled	<input checked="" type="checkbox"/>
geometry	[(0, 0), 526 x 409]
X	0
Y	0
Width	526
Height	409
sizePolicy	[Preferred, Preferred, ...]
Horizontal Policy	Preferred



# 2011 Umyeon Mt. landslide



- 140 occurrence (blue points)
- 31 sub-catchment (yellow lines)

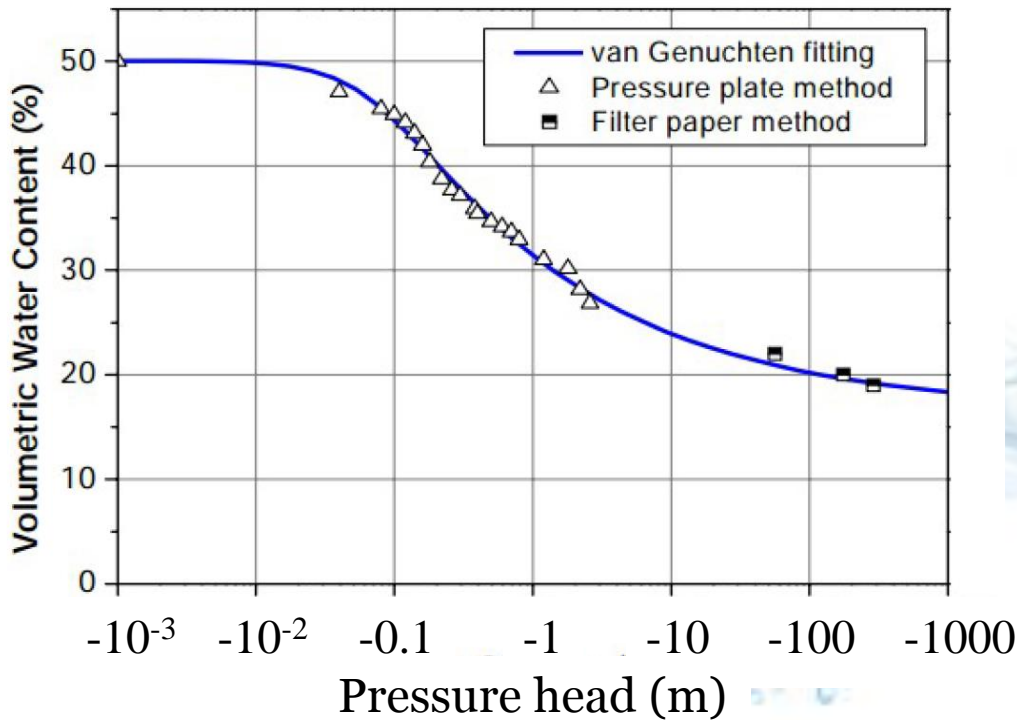
Table 1. Parameter setting

$K_s$	$\gamma_s$	$\phi$	$c_s$	$c_r$	$\theta_s$	$\theta_r$
(m/s)	(kg/m <sup>3</sup> )	(°)	(kg/m <sup>2</sup> )	(kg/m <sup>2</sup> )	(m <sup>3</sup> /m <sup>3</sup> )	(m <sup>3</sup> /m <sup>3</sup> )
1.3E-05	2195	29.63	1838	0	0.5	0.18

# Parameters

Table 1. Parameter setting

$K_s$	$\gamma_s$	$\varphi$	$c_s$	$c_r$	$\theta_s$	$\theta_r$
(m/s)	(kg/m <sup>3</sup> )	(°)	(kg/m <sup>2</sup> )	(kg/m <sup>2</sup> )	(m <sup>3</sup> /m <sup>3</sup> )	(m <sup>3</sup> /m <sup>3</sup> )
1.3E-05	2195	29.63	1838	0	0.5	0.18



$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = \left\{ \frac{1}{1 + (\alpha |\psi|^{n_v})} \right\}^{1-1/n_v}$$

$$K = K_s S_e^{1/2} \left\{ 1 - \left( 1 - S_e^{n_v/(n_v-1)} \right)^{1-1/n_v} \right\}^2$$

$$\theta_s = 0.5, \quad \theta_r = 0.18,$$

$$\alpha = 1.0 \text{m}^{-1}, \quad n_v = 3.0$$

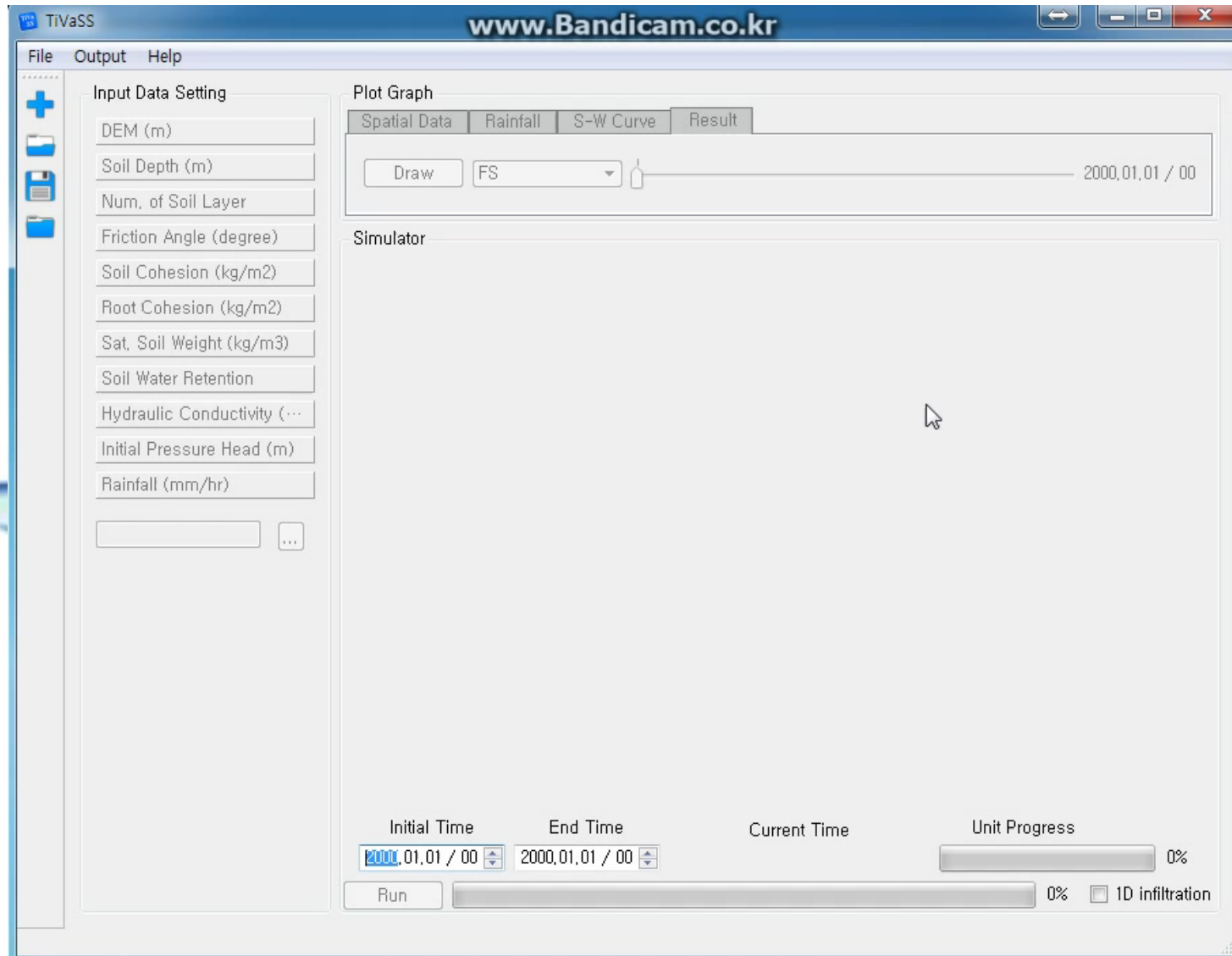
# TiVaSS (Time-Variant Slope Stability model)

The screenshot shows a Windows File Explorer window with the address bar set to 'Dropbox > prog > TiVaSS\_release'. The window title is 'www.Bandicam.co.kr'. The search bar contains 'TiVaSS\_release 검색'. The main pane displays a list of files and folders with columns for '이름' (Name), '수정한 날짜' (Modified Date), '유형' (Type), and '크기' (Size). The 'TIVASS.exe' file is highlighted.

이름	수정한 날짜	유형	크기
ex_data	2016-05-24 오후...	파일 폴더	
iconengines	2016-03-31 오후...	파일 폴더	
imageformats	2016-03-31 오후...	파일 폴더	
platforms	2016-03-31 오후...	파일 폴더	
printsupport	2016-03-31 오후...	파일 폴더	
translations	2016-03-31 오후...	파일 폴더	
libEGL.dll	2015-10-13 오전...	응용 프로그램 확장	21KB
libgcc_s_dw2-1.dll	2014-12-22 오전...	응용 프로그램 확장	118KB
libGLESV2.dll	2015-10-13 오전...	응용 프로그램 확장	2,240KB
libstdc++-6.dll	2014-12-22 오전...	응용 프로그램 확장	1,003KB
libwinpthread-1.dll	2014-12-22 오전...	응용 프로그램 확장	48KB
opengl32sw.dll	2014-09-23 오후...	응용 프로그램 확장	14,864KB
Qt5Core.dll	2016-03-31 오후...	응용 프로그램 확장	5,265KB
Qt5Gui.dll	2015-10-13 오전...	응용 프로그램 확장	5,210KB
Qt5PrintSupport.dll	2015-10-13 오전...	응용 프로그램 확장	350KB
Qt5Svg.dll	2015-10-13 오전...	응용 프로그램 확장	324KB
Qt5Widgets.dll	2015-10-13 오전...	응용 프로그램 확장	6,389KB
TIVASS.exe	2016-04-28 오후...	응용 프로그램	1,270KB
TiVaSS_release.zip	2016-05-11 오후...	압축(ZIP) 파일	16,929KB

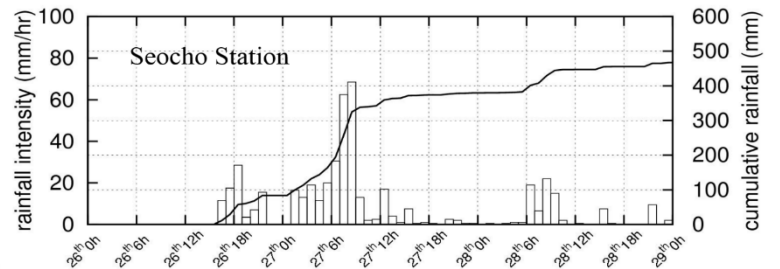
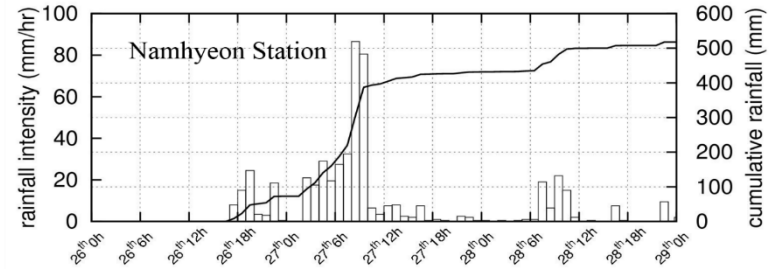
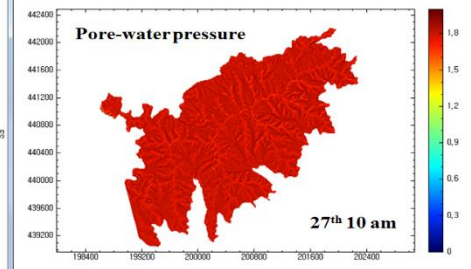
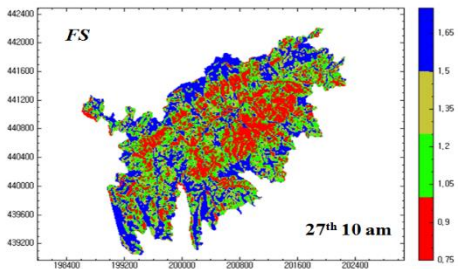
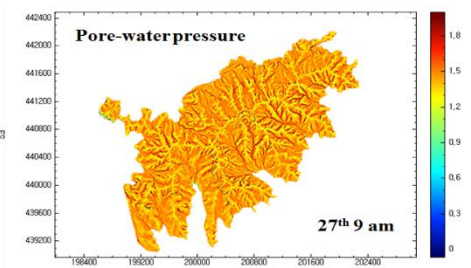
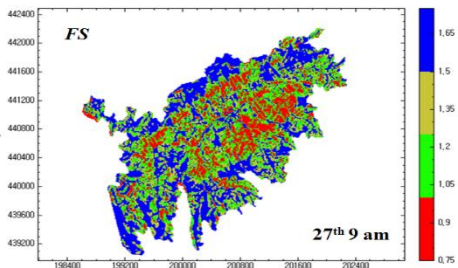
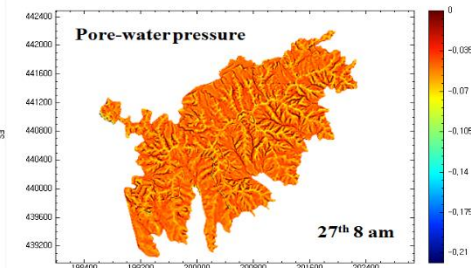
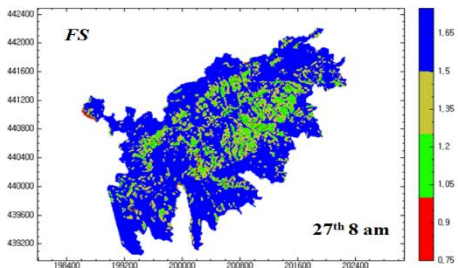
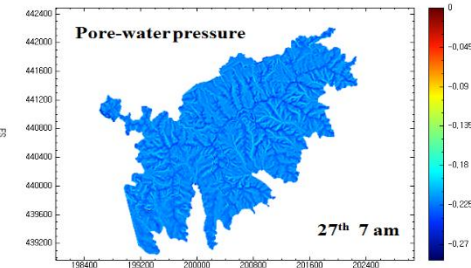
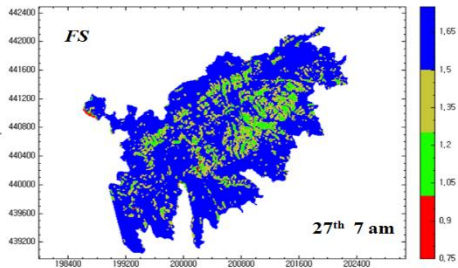
At the bottom of the window, a status bar shows details for the selected file: 'VASS.exe 수정한 날짜: 2016-04-28 오후 5:37 만든 날짜: 2016-03-31 오후 11:24 용 프로그램 크기: 1.23MB'.

# Visualization of Results

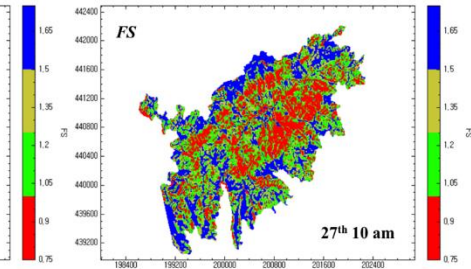
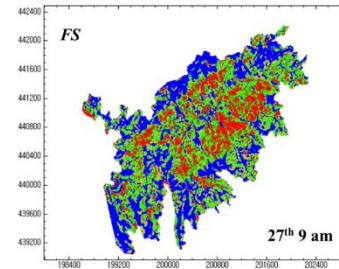
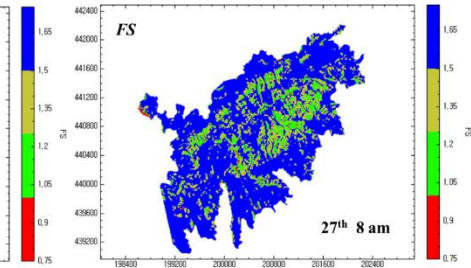
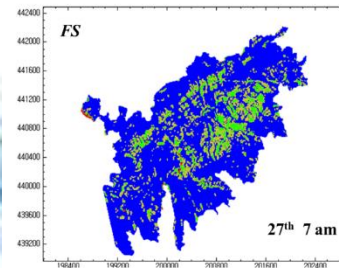


# Simulation Results

## 3D subsurface flow



## 1D subsurface flow



# Accuracy assessment (3D subsurface flow)

- The accuracy index ( $LR_{class}$ ) is proposed by Park et al. (2013)
- It can be used when **the landslide area is unknown**.

FS classes	Slope failure location (a)	% of location (c)=a/b	% of predicted area (d)	$LR_{class}$ (e)=c/d	% of $LR_{class}$ =e/f
FS < 1	95	67.86	19.25	3.52	72.54
1 < FS < 1.25	18	12.86	20.81	0.61	12.72
1.25 < FS < 1.5	12	8.57	18.79	0.46	9.39
1.5 < FS	15	10.71	41.15	0.26	5.36
Sum	140 (b)	100	100	4.86 (f)	100

# Accuracy assessment (1D subsurface flow)

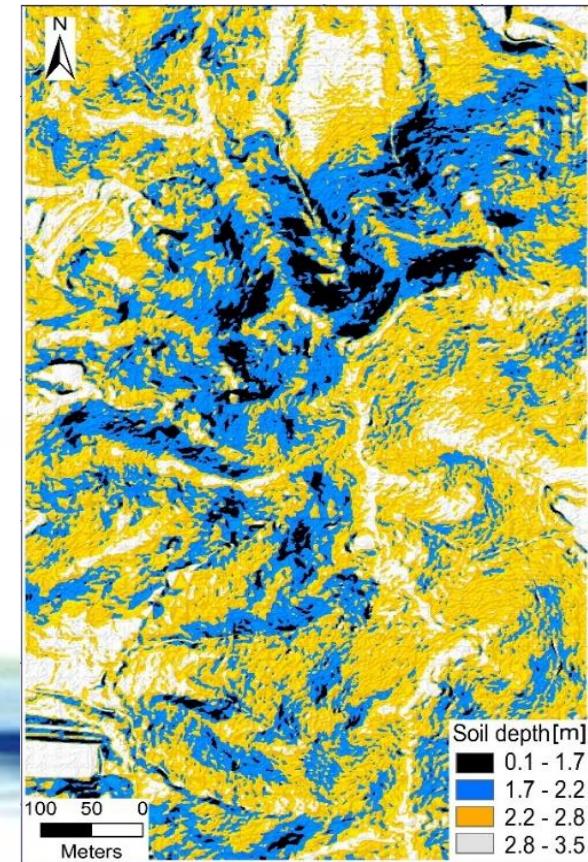
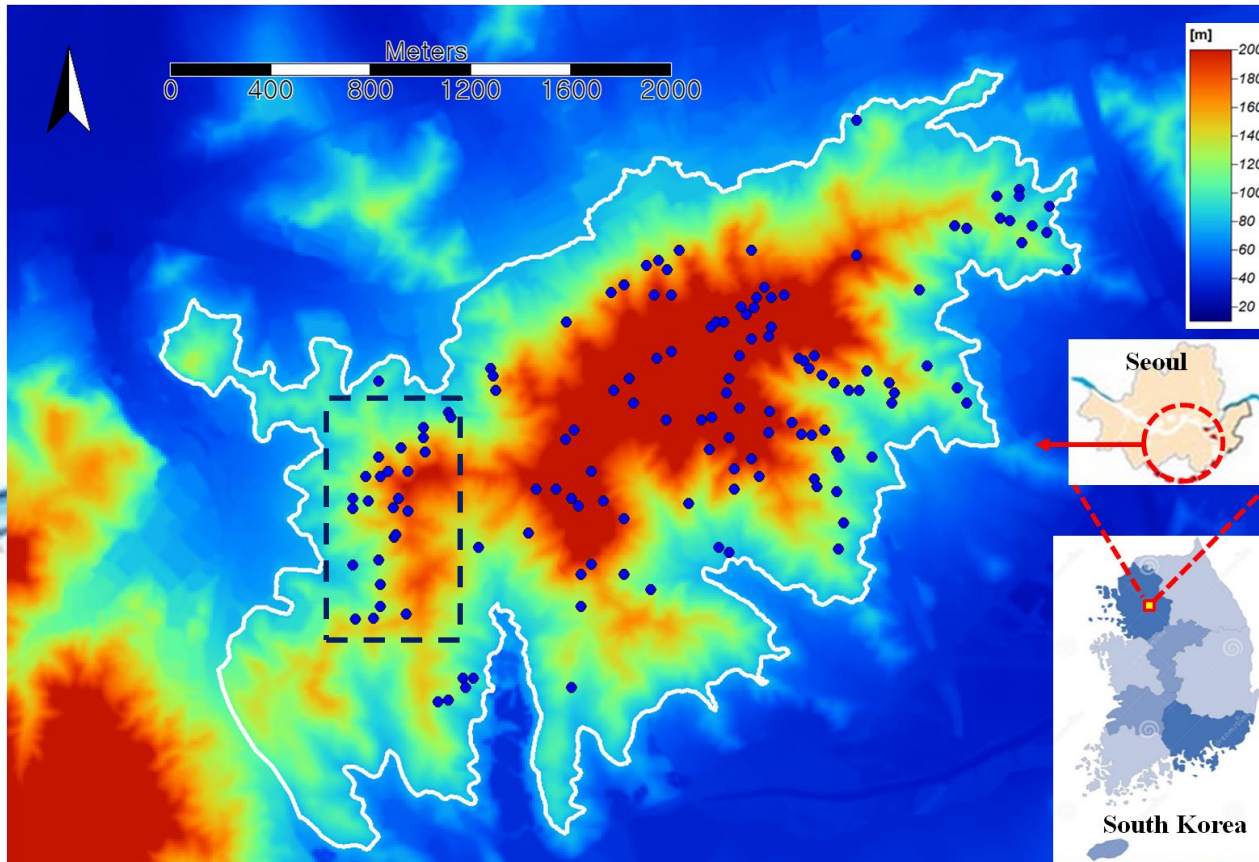
- The accuracy index ( $LR_{class}$ ) is proposed by Park et al. (2013)
- It can be used when **the landslide area is unknown**.

FS classes	Slope failure location (a)	% of location (c)=a/b	% of predicted area (d)	$LR_{class}$ (e)=c/d	% of $LR_{class}$ =e/f
FS < 1	93	66.43	19.26	3.45	70.81
1 < FS < 1.25	20	14.29	20.88	0.68	14.05
1.25 < FS < 1.5	13	9.29	18.81	0.49	10.14
1.5 < FS	14	10.0	40.06	0.24	5.00
Sum	140 (b)	100	100	4.87 (f)	100

# TiVaSS vs TRIGRS

1 X 1 m DEM

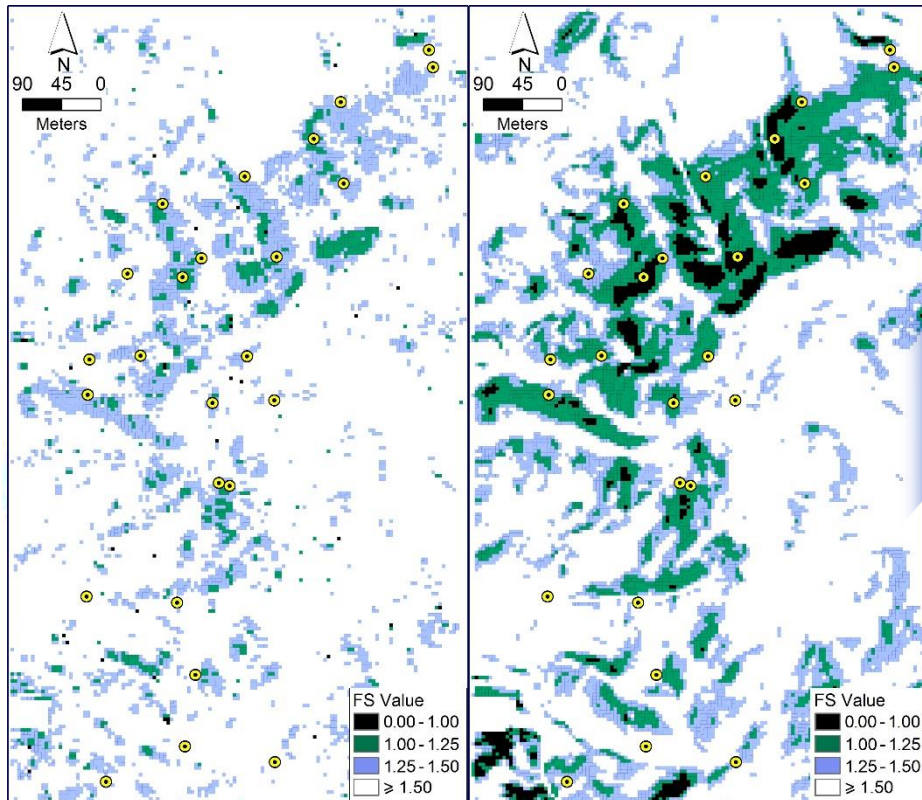
Distributed soil depth



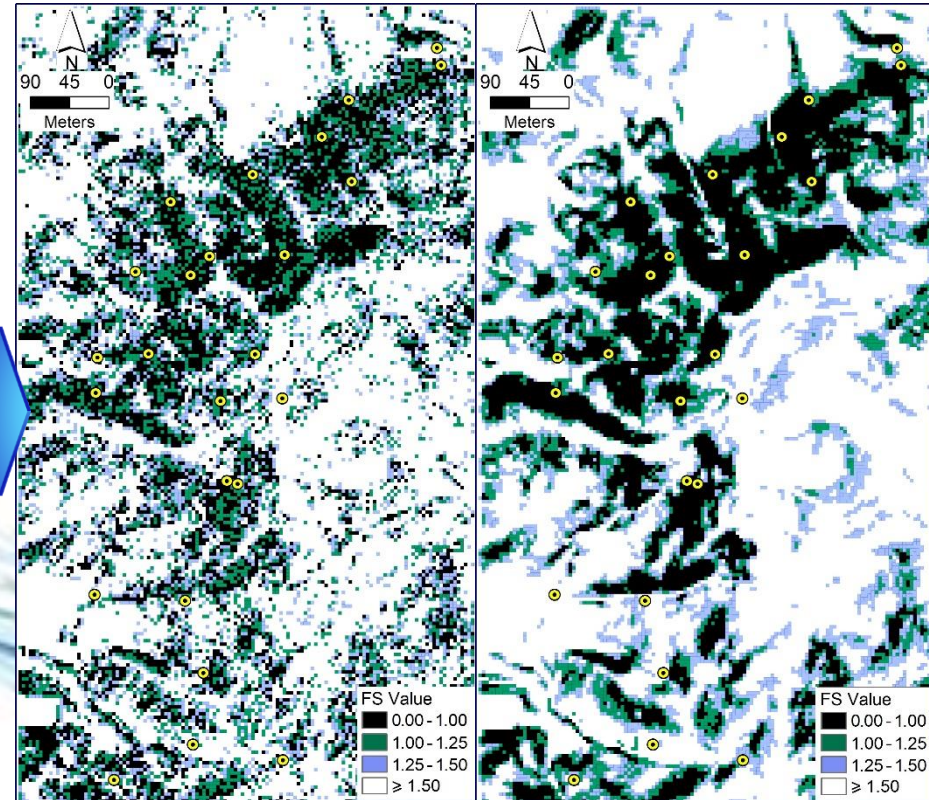
# Results, FS

## TiVaSS vs TRIGRS

## TiVaSS vs TRIGRS



6 am July 27



9 am July 27

# Accuracy assessment TiVaSS vs TRIGRS

9 am July 27

FS Classes	Observed sliding sites		% Observed sliding sites		% of predicted area		$LR_{class}$		% $LR_{class}$	
	TRIGRS	TiVaSS	TRIGRS	TiVaSS	TRIGRS	TiVaSS	TRIGRS	TiVaSS	TRIGRS	TiVaSS
FS <1	15	16	60.0	64.0	19.3	20.1	3.11	3.19	64.18	65.75
1-1.25	2	4	8.0	16.0	12.8	15.0	0.63	1.07	12.94	22.01
1.25-1.50	3	1	12.0	4.0	16.7	14.3	0.72	0.28	14.81	5.73
> 1.50	5	4	20.0	16.0	51.2	50.6	0.39	0.31	8.07	6.51
Sum	25	25	100.0	100.0	100.0	100.0	4.85	4.85	100.0	100

# Accuracy assessment TiVaSS vs TRIGRS

6 am July 27

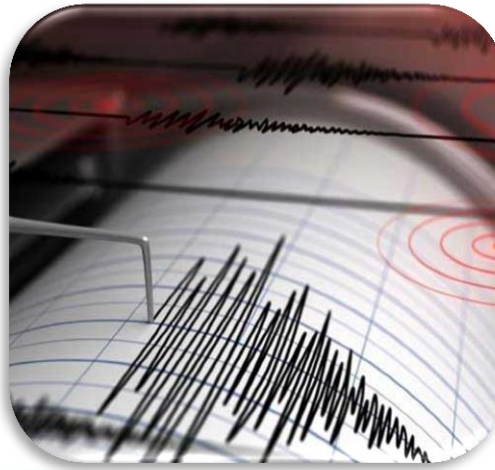
FS Classes	Observed sliding sites		% Observed sliding sites		% of predicted area		$LR_{class}$		% $LR_{class}$	
	TRIGRS	TiVaSS	TRIGRS	TiVaSS	TRIGRS	TiVaSS	TRIGRS	TiVaSS	TRIGRS	TiVaSS
FS <1	4	0	16.0	0.0	3.1	0.1	5.16	0.00	56.27	0.0
1-1.25	10	1	40.0	24.0	14.4	3.1	2.79	1.30	30.38	27.60
1.25-1.50	4	11	16.0	32.0	20.7	15.9	0.77	2.77	8.42	58.75
> 1.50	7	13	28.0	44.0	61.8	80.9	0.45	0.64	4.93	13.65
Sum	25	25	100.0	100.0	100.0	100.0	9.17	4.71	100.0	100.0

# Summary

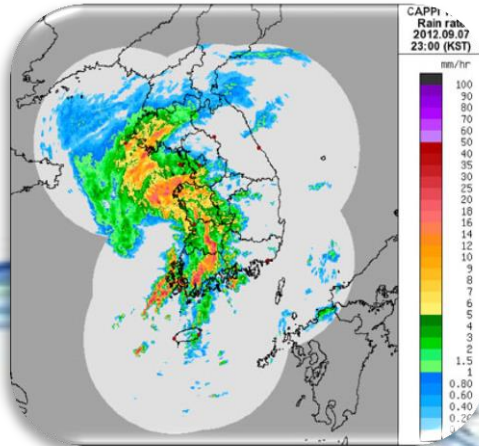
- **TiVaSS** (Time-Variant Slope Stability model) was developed based on the combination of the infinite slope stability model and the 3D subsurface flow model
- TiVaSS was applied to 2011 Umyeon Mt. landslide event and the performance was satisfactory in overall.
- TiVaSS works on Windows OS, and supports GUI tool and basic 2D data visualization tool.
- Updating and further application is ongoing...
- <https://sites.google.com/site/cnuaehelab/software/tivass>

# Future Plan

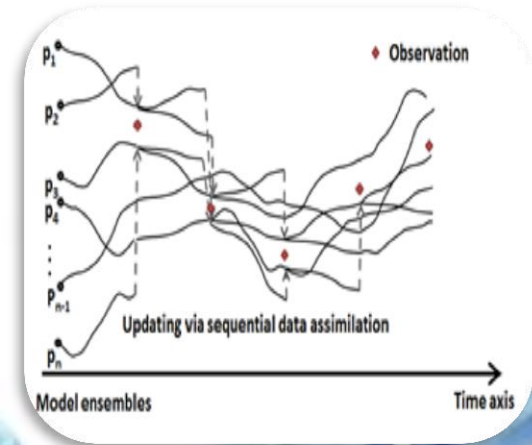
## Earthquake



## Distributed rainfall gauged by Radar




## Data assimilation for forecasting

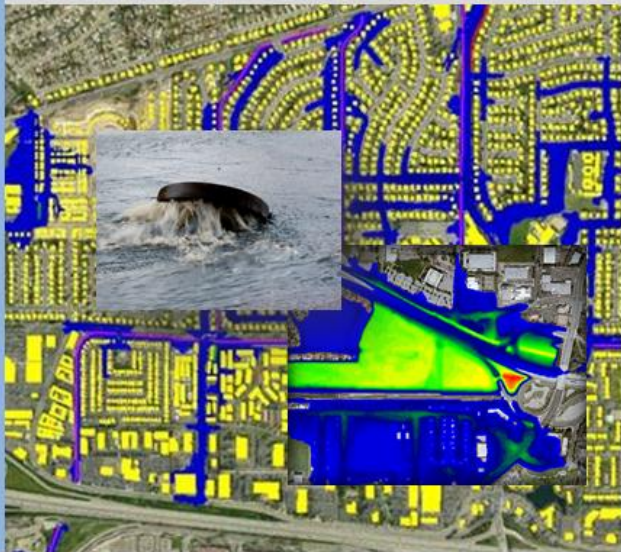


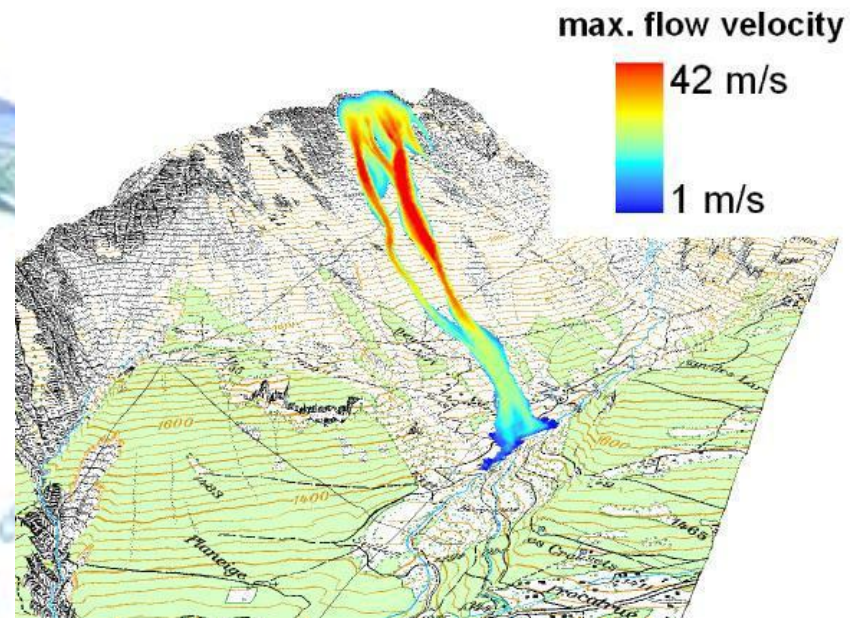
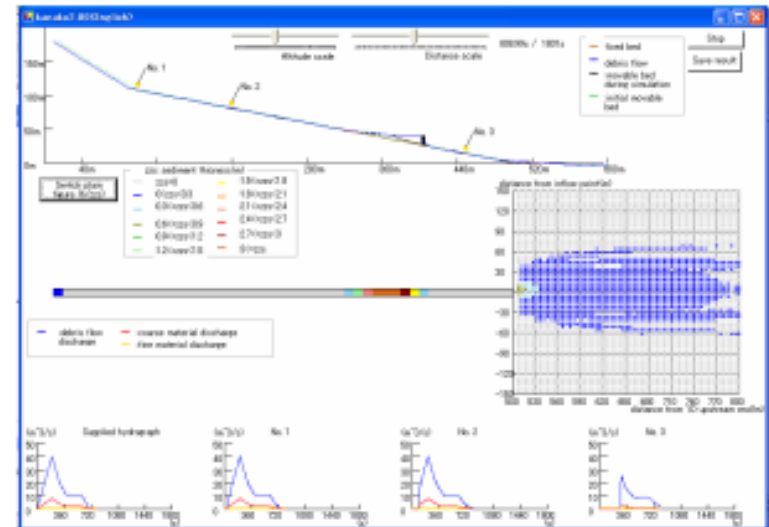
# Development of Debris flow model

Training Package



FLO-2D Tutorials





# Governing Equation

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} = \mathbf{s},$$

$$\mathbf{q} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} hu \\ hu^2 + gh^2 / 2 \\ huv \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} hv \\ huv \\ hv^2 + gh^2 / 2 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} E \\ -hgz_x - S_{fx} \\ -hgz_y - S_{fy} \end{pmatrix},$$

$h$  : water depth,  $u, v$  : velocity,

$E$  : mass source of erisional and deposition effect,

$S_{fx}, S_{fy}$  : friction effect of debris flow mixture,

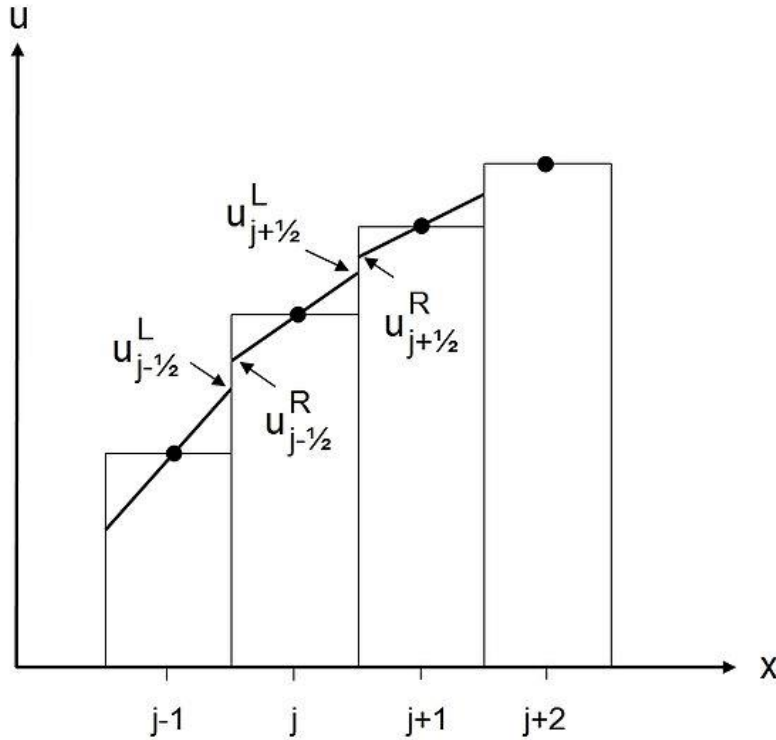
$$S_{fx} = n_u \left[ \mu gh + \frac{g |\vec{u}|^2}{\xi} \right], S_{fy} = n_v \left[ \mu gh + \frac{g |\vec{u}|^2}{\xi} \right]$$

$$|\vec{u}|^2 = |u^2 + v^2|$$

Voellmy friction model



# MUSCL scheme



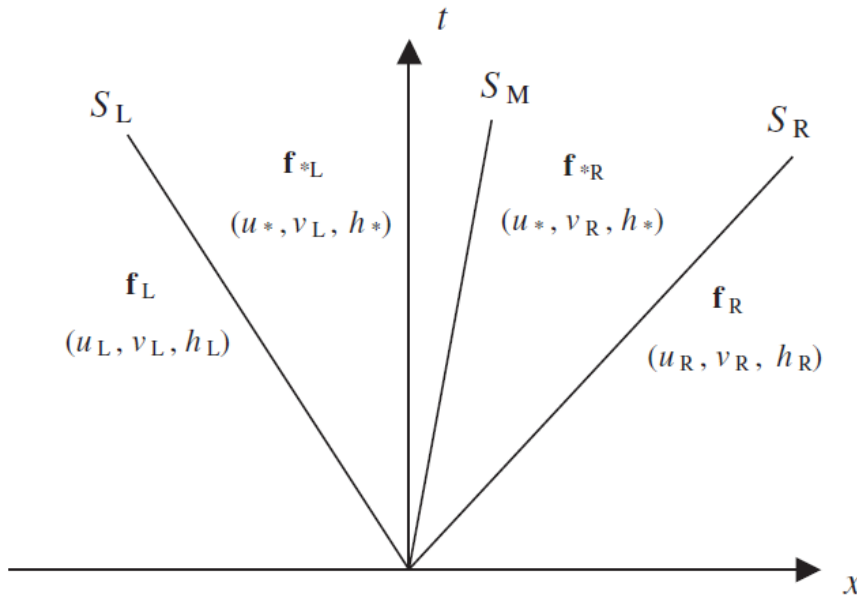
Piecewise linear extrapolation

$$\mathbf{q}_{i,j,f} = \mathbf{q}_{i,j} + \mathbf{r}_f \cdot \nabla \mathbf{q}_{i,j}$$

$$\nabla \mathbf{q}_x = \psi \left( \frac{\mathbf{q}_{j+1} - \mathbf{q}_j}{x_{j+1} - x_j}, \frac{\mathbf{q}_j - \mathbf{q}_{j-1}}{x_j - x_{j-1}} \right)$$

$\psi(a,b)$  : limiter function.

# HLLC Reimann solver



$$\mathbf{f}_{i+1/2,j} = \begin{cases} \mathbf{f}_L & \text{if } 0 \leq S_L \\ \mathbf{f}_{*L} & \text{if } S_L \leq 0 \leq S_M \\ \mathbf{f}_{*R} & \text{if } S_M \leq 0 \leq S_R \\ \mathbf{f}_R & \text{if } 0 \geq S_R \end{cases}$$

$$S_L = \begin{cases} u_R - 2\sqrt{gh_R} & \text{if } h_L = 0 \\ \min(u_L - \sqrt{gh_L}, u_* - \sqrt{gh_*}) & \text{if } h_L > 0 \end{cases}$$

$$S_R = \begin{cases} u_L + 2\sqrt{gh_L} & \text{if } h_R = 0 \\ \max(u_R + \sqrt{gh_R}, u_* + \sqrt{gh_*}) & \text{if } h_R > 0 \end{cases}$$

$$S_M = \frac{S_L h_R (u_R - S_R) - S_R h_L (u_L - S_L)}{h_R (u_R - S_R) - h_L (u_L - S_L)}$$

$$h_* = \frac{1}{g} \left[ \frac{1}{2} (\sqrt{gh_L} + \sqrt{gh_R}) + \frac{1}{4} (u_L - u_R) \right]^2$$

$$u_* = \frac{1}{2} (u_L + u_R) + \sqrt{gh_L} - \sqrt{gh_R}$$



# Temporal discretisation

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \mathbf{s},$$

$$A_{i,j} \left. \frac{\partial \mathbf{q}}{\partial t} \right|_{i,j} + F_{i,j} = S_{i,j}$$

Prediction step

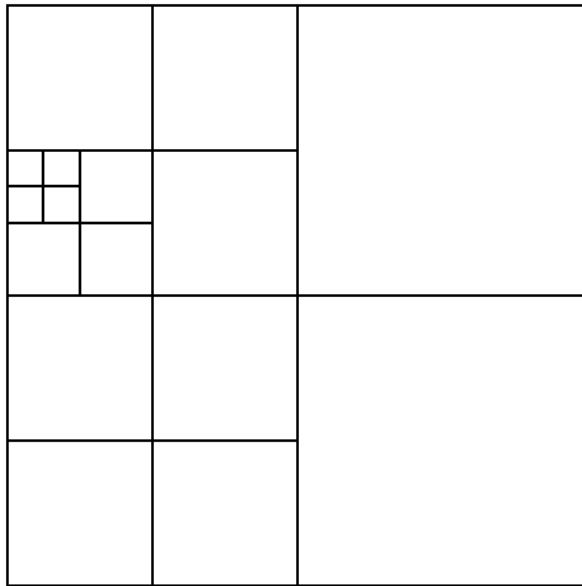
$$A_{i,j} \mathbf{q}_{i,j}^{n+1/2} = A_{i,j} \mathbf{q}_{i,j}^n - \frac{\Delta t}{2} (F_{i,j}^n + S_{i,j}),$$

Correction step

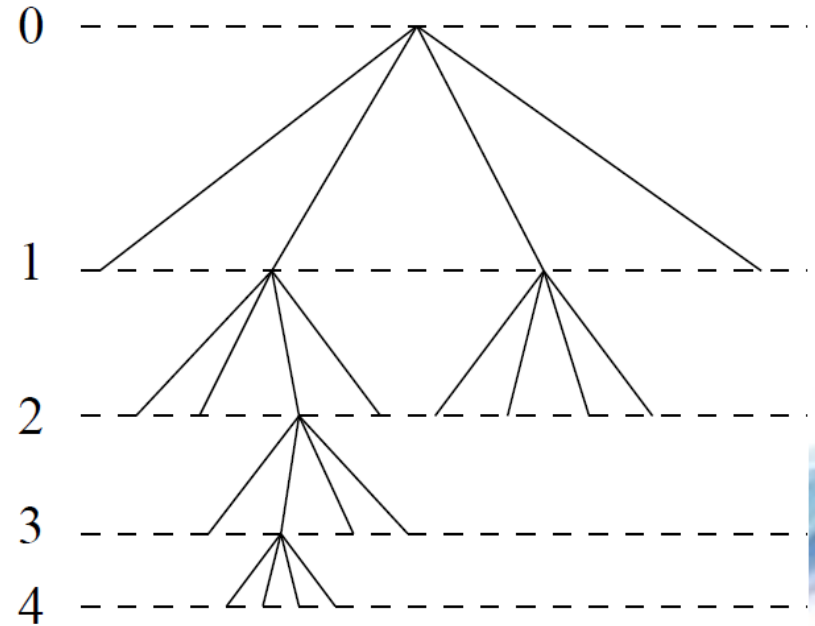
$$A_{i,j} \mathbf{q}_{i,j}^{n+1} = A_{i,j} \mathbf{q}_{i,j}^n - \Delta t (F_{i,j}^{n+1/2} + S_{i,j}).$$



# Quadtree grid



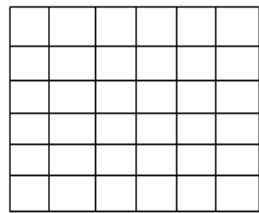
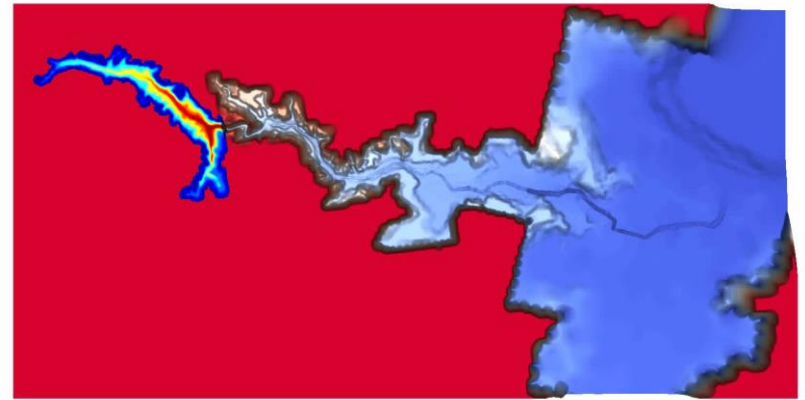
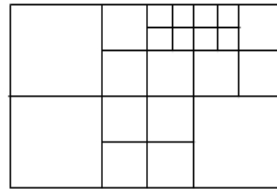
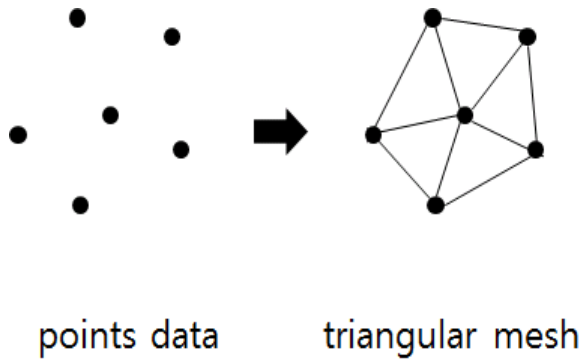
Adaptive mesh



Quadtree grid structure



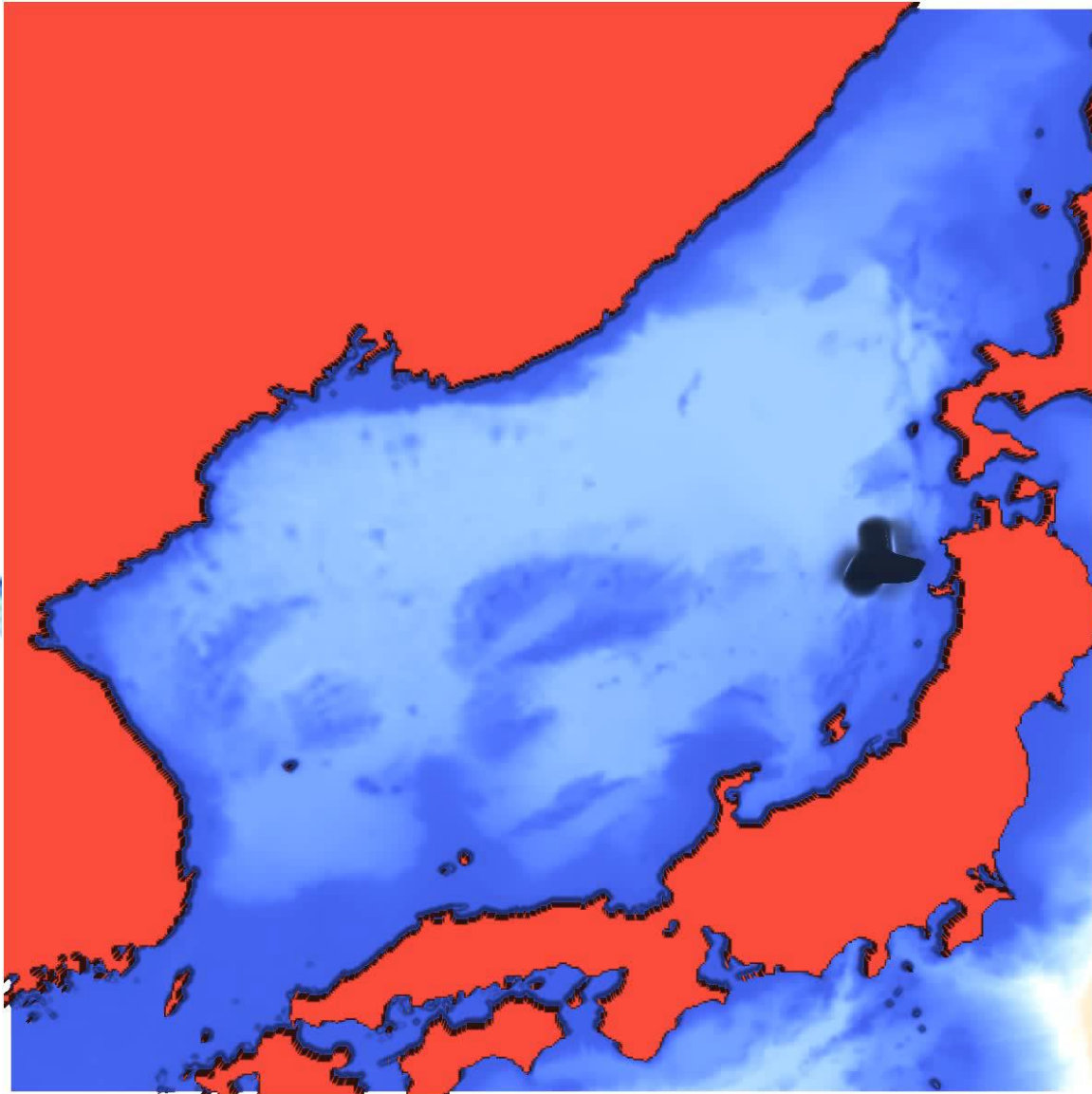
# Adaptive Mesh Generation



quadtree adaptive mesh generation



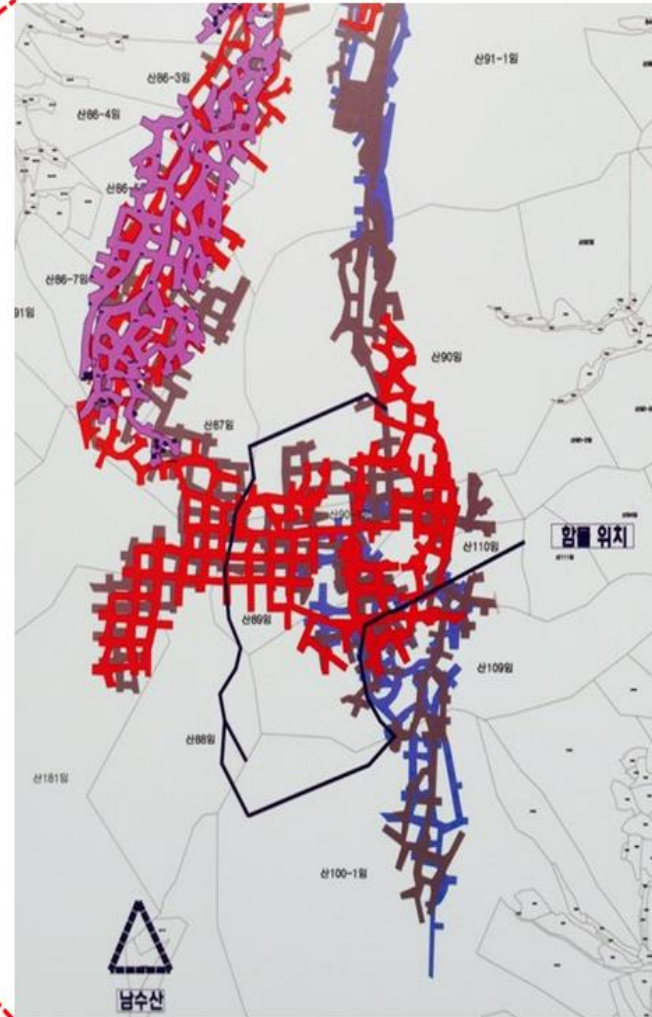
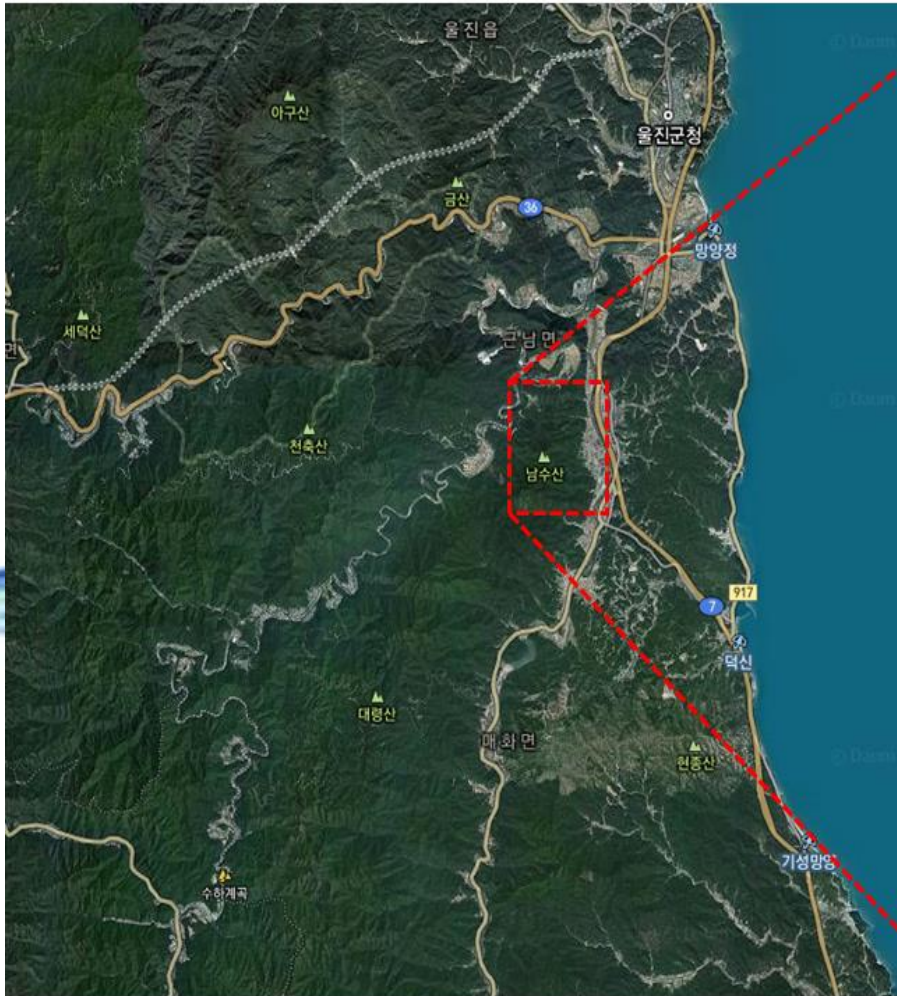
# Adaptive Mesh Generation



- 1983 tsunami
- 7.5 m maximum wave height
- 5 dead persons
- More than 400 refugees



# Study Area

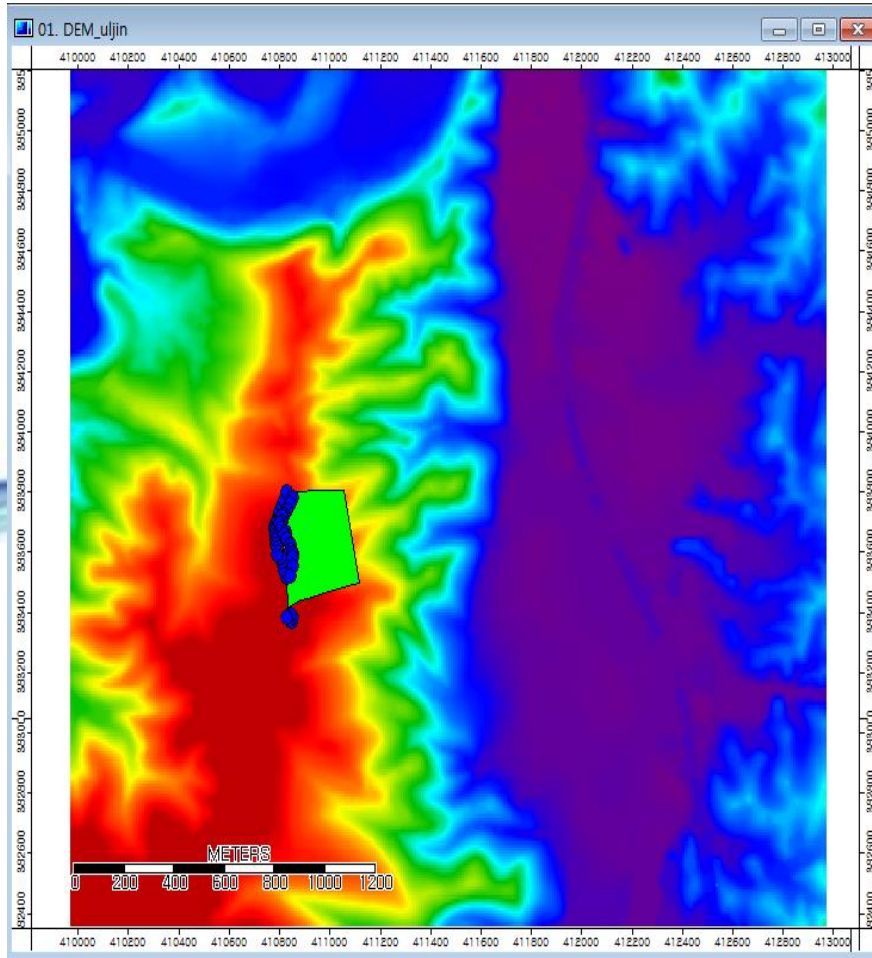


● Uljin-goon, Meahwa-myeon

# Study Area, Crack

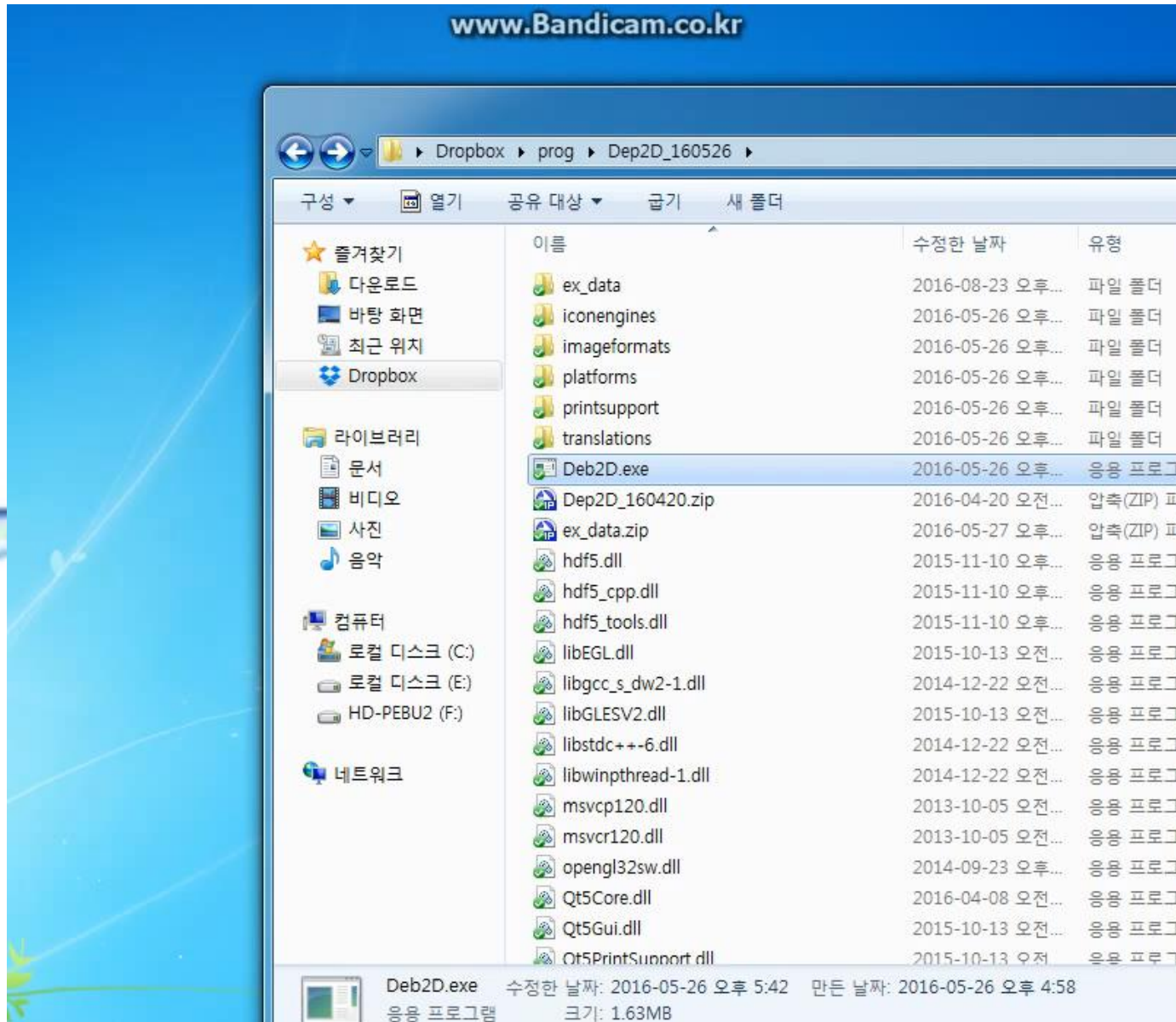


# Initial condition



- 3m uniform depth
- 10m uniform depth

# Deb2D (Debris flow model, 2D)



# Visualization of Results

www.Bandicam.co.kr

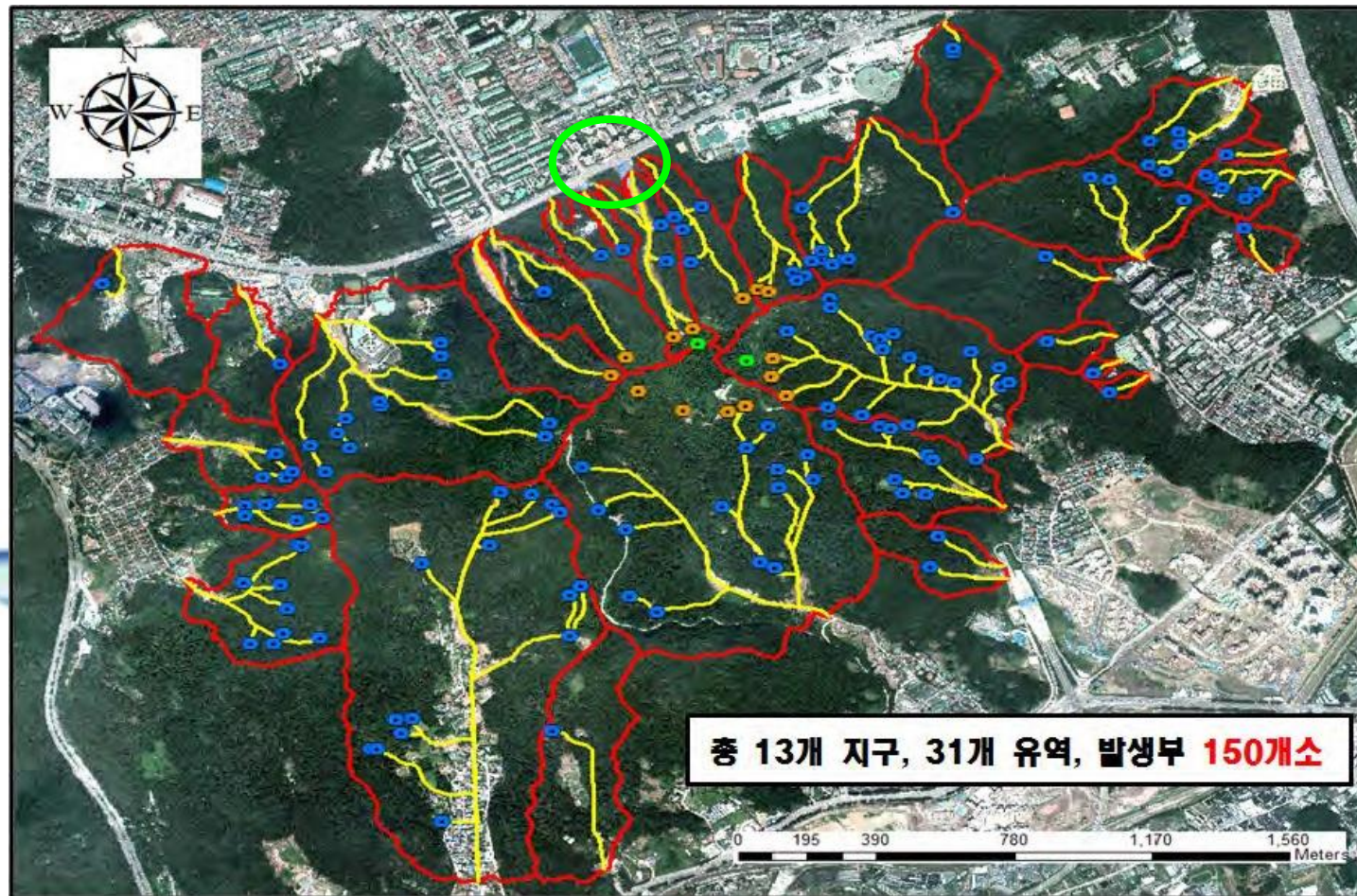
Dropbox > prog > Dep2D\_160526

구성 열기 공유 대상 급기 새 폴더

즐거찾기	이름	수정한 날짜	유형
다운로드	ex_data	2016-08-23 오후...	파일 폴더
바탕 화면	iconengines	2016-05-26 오후...	파일 폴더
최근 위치	imageformats	2016-05-26 오후...	파일 폴더
Dropbox	platforms	2016-05-26 오후...	파일 폴더
라이브러리	printsupport	2016-05-26 오후...	파일 폴더
문서	translations	2016-05-26 오후...	파일 폴더
비디오	Deb2D.exe	2016-05-26 오후...	응용 프로그램
사진	Dep2D_160420.zip	2016-04-20 오전...	압축(ZIP) 파일
음악	ex_data.zip	2016-05-27 오후...	압축(ZIP) 파일
컴퓨터	hdf5.dll	2015-11-10 오후...	응용 프로그램
로컬 디스크 (C:)	hdf5_cpp.dll	2015-11-10 오후...	응용 프로그램
로컬 디스크 (E:)	hdf5_tools.dll	2015-11-10 오후...	응용 프로그램
HD-PEBU2 (F:)	libEGL.dll	2015-10-13 오전...	응용 프로그램
네트워크	libgcc_s_dw2-1.dll	2014-12-22 오전...	응용 프로그램
	libGLESV2.dll	2015-10-13 오전...	응용 프로그램
	libstdc++-6.dll	2014-12-22 오전...	응용 프로그램
	libwinpthread-1.dll	2014-12-22 오전...	응용 프로그램
	msvcp120.dll	2013-10-05 오전...	응용 프로그램
	msvcr120.dll	2013-10-05 오전...	응용 프로그램
	opengl32sw.dll	2014-09-23 오후...	응용 프로그램
	Qt5Core.dll	2016-04-08 오전...	응용 프로그램
	Qt5Gui.dll	2015-10-13 오전...	응용 프로그램
	Qt5PrintSupport.dll	2015-10-13 오전...	응용 프로그램

Dep2D.exe 수정한 날짜: 2016-05-26 오후 5:42 만든 날짜: 2016-05-26 오후 4:58  
응용 프로그램 크기: 1.63MB

# Case study: 2011 Umyeon Mt. debris flow



- 150 occurrence (blue points)
- 31 water shed (yellow lines)

# Case study: 2011 Umyeon Mt. debris flow

- Average slope : 19 degree
- watershed area : 75600 m<sup>2</sup>
- **3 fatality**
- Reamian APT is **damaged up to 3<sup>rd</sup> floor**

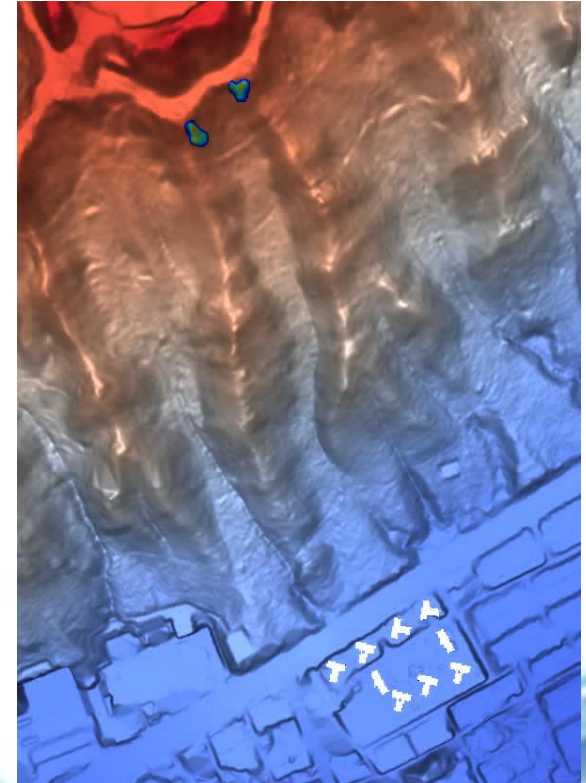
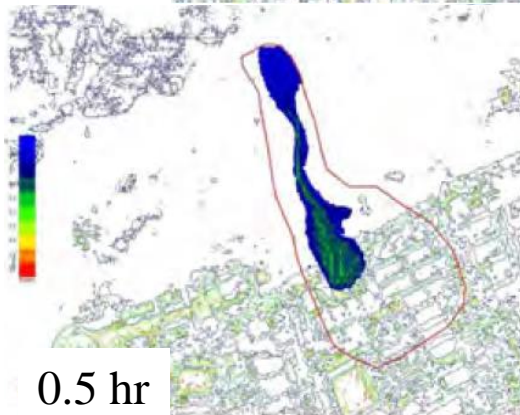
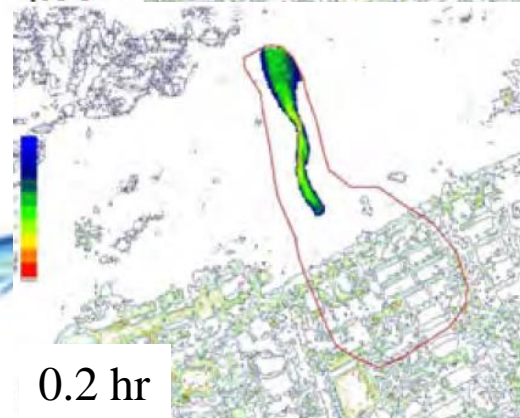


# Case study: 2011 Umyeon Mt. debris flow



정치 광복회, 日 의원 울릉도 방문 반대 집회

# Simulation Result : comparison with FLO-2D



**FLO-2D**  
erosion (X)  
Buildings (X)

**Developed model**  
Erosion (O)  
Buildings (O)

**Observed data**  
Black box → guessed velocity **27 m/s**  
Flow depth at APT → **3<sup>rd</sup> floor**  
**well agreed**

# Summary

- Adaptive grid based Debris flow model, **Deb2D**, is developed.
- A simple GUI and visualization tools are supported in Deb2D.
- 2011 Umyeon Mt. debris flow seems to be successfully simulated by the developed model although further verification is required
- Model verification is ongoing.
- <https://sites.google.com/site/cnuaehelab/software/dep2d>

**Thank you !!**



**충남대학교**  
CHUNGNAM NATIONAL UNIVERSITY

# Balance issue

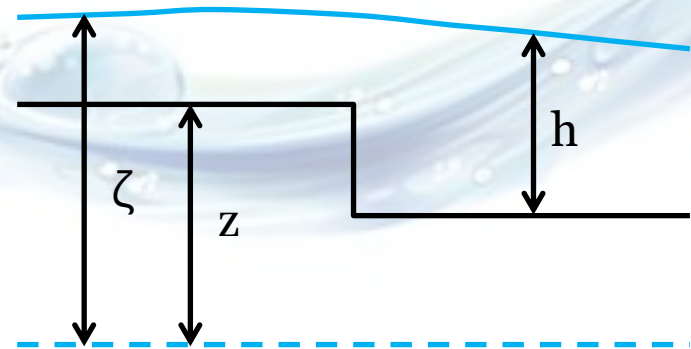
$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \mathbf{s},$$

$$\mathbf{q} = \begin{pmatrix} h \\ hu \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} hu \\ hu^2 + gh^2 / 2 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 0 \\ -hgz_x \end{pmatrix},$$

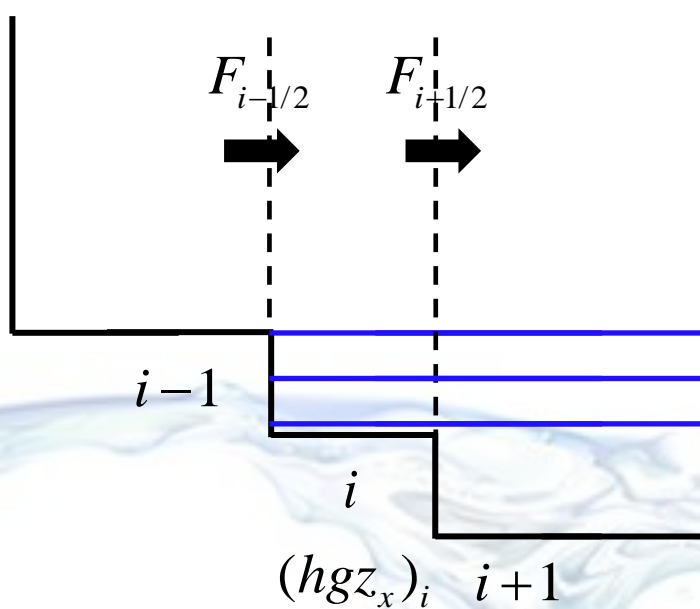
A **hydrostatic solution** is given as

$$u = 0, \quad h + z = \text{constant}$$

$$\frac{\partial}{\partial x} \left( gh^2 / 2 \right) + hg \frac{\partial z}{\partial x} = 0$$



# Well-balanced scheme



$$\frac{\partial}{\partial x} \left( gh^2 / 2 \right) + hg \frac{\partial z}{\partial x} = 0$$

$$\parallel \qquad \parallel$$

$$(F_{i+1/2} - F_{i-1/2}) + hg (z_x)_i = 0$$

const	const	Continuous
non-const	const	Discretized

“hydrostatic reconstruction” technique  
(Audusse et al., 2004)

Well-balanced scheme for shallow water equation

(i.e., Rogers et al., 2003; Audusse et al., 2004; Begnudelli & Sanders, 2006; Benkhaldoun et al., 2007; Caleffi et al., 2007; Gallardo et al. 2007; Liange & Borthwick, 2009; Liang & Marche, 2009; etc)

# Friction (flow resistance) models

Flow resistance relation	Flow resistance term $S_f$
Full Bingham	$S_f = \frac{\tau_0}{\rho gh}$ $\tau_0 \text{ can be determined by: } 2\tau_0^3 - 3\left(\tau_y + 2\frac{\mu_B q}{h^2}\right)\tau_0^2 + \tau_y^3 = 0$
Simplified Bingham	$S_f = \frac{\tau_0}{\rho gh} \text{ with } \tau_0 = 1.5\tau_y + 3\frac{\mu_B q}{h^2}$
Voellmy	$S_f = \frac{q\sqrt{q^2}}{h^2 C^2 h_r} + \cos \alpha \tan \delta$
Turbulent & Coulomb	$S_f = \frac{n^2 q \sqrt{q^2}}{h^2 h_r^{4/3}} + \cos \alpha \tan \delta$
Turbulent & Yield	$S_f = \frac{n^2 q \sqrt{q^2}}{h^2 h_r^{4/3}} + \frac{\tau_y}{\rho gh}$
Turbulent, Coulomb & yield	$S_f = \frac{n^2 q \sqrt{q^2}}{h^2 h_r^{4/3}} + \frac{\tau_i}{\rho gh}$ <p>with <math>\tau_i = \min(\tau_y ; \rho gh \cos \alpha \tan \delta)</math></p>
Quadratic	$S_f = \frac{n^2 q \sqrt{q^2}}{h^2 h_r^{4/3}} + \frac{\kappa \eta q}{8h^3 \rho g} + \frac{\tau_y}{\rho gh}$
Coulomb viscous	Full Bingham with $\tau_y = \rho gh \cos \alpha \tan \delta$

